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Exam Two , MTH 221 , Fall 2019

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SCORE = $\frac{45}{45}$

QUESTION 1. (10 points)

Imagine the following setting:
You are "Mr/Ms know it". Your friend just sent you the following messages on "WhatsApp"

(i) Hi. My instructor today was talking about Range(T), where T is some linear transformation. My question: Assume $Range(T) = span\{(3,0,1),(-3,1,1)\}$ What does it mean that $Q = (a_1, a_2, a_3) \in Range(T)$?

Answer (Clearly):

Enough

it means that (a_1, a_2, a_3) is dependent on/can be written as a linear combination of $(3,0,1)$ and $(-3,1,1)$ ~~so it's in range(T) (there exists value α_1, α_2 $\alpha_1(3,0,1)+\alpha_2(3,1,1) = (a_1,a_2,a_3)$ other than 0)~~

(ii) Thanks, one more please: Assume $T : R^3 \to R^3$ is a linear transformation such that $T(a_1, a_2, a_3) = (a_1 + a_2 + a_3, -a_2 + a_3)$. What is the standard matrix presentation M of T?

Answer (Clearly) $M =$

$$\begin{array}{ccc} a_1 & a_2 & a_3 \end{array}$$
$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

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(iii) Appreciate it. Let $T : R^3 \to R$ be a linear transformation such that $T(1,2,1) = 5$ and $T(1,0,1) = -7$. What is $T(0,2,0)$?

Answer (Clearly):

$(0,2,0) = c_1(1,2,1) + c_2(1,0,1)$

$c_1 + c_2 = 0$ $\quad c_1 = 1$

$2c_1 = 2$ $\quad c_2 = -1$

$c_1 + c_2 = 0$

$T(0,2,0) = 1\,T(1,2,1) + -1\,T(1,0,1)$

$T(0,2,0) = 5 - (-7) = 12$

(iv) Given $M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$ is the standard matrix presentation of a linear transformation $T : R^3 \to R^3$. One student said "ooo" by staring, I can conclude that $(0,0,\pi) \in Z(T)$. How did he know that just by staring?

Answer(Clearly):

it means $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \pi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$0\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \pi\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

he knew that because the 3rd column has all 0's so no matter $(0,0,x)$ you multiply you'll get an image of $(0,0,0)$ and that pt $\in Z(T)$

(v) Let $T : R^3 \to R^3$ be a linear transformation such that T is onto. The instructor said "it is clear that $Z(T) = \{(0,0,0)\}$". It is not so clear to me, can you tell me why?

onto means dim(range) = dim(codomain) = 3

and we know that dim(range) + dim(Z) = dim(domain)

$3 + X = 3$

dim(Z) (X) has to be 0 then which means the only pt in Z(T) should be (0,0,0)

QUESTION 2. (5 points) Given $B = \{x + x^2 + x^3, x + x^3, f_3, f_4, f_5\}$ is a basis for P_5. Find a possibility for the polynomials f_3, f_4 and f_5. (show the work, but briefly)

$B' = [(0,1,1,1,0), (0,1,0,1,0), Q_3, Q_4, Q_5]$ 08

$f_3 = x^4$

$f_4 = x$

$f_5 = 1$

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{-R_1+R_2 \to R_2} \begin{array}{c} \\ \\ Q_1 \leftarrow \\ Q_2 \leftarrow \\ Q_3 \leftarrow \end{array} \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

QUESTION 3. (6 points)

(i) Convince me that $F = \{A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid 0 \le a + d \le 3\}$ is not a subspace of $R^{2\times 2}$ (i.e., F consists of all matrices 2×2 so that the sum of the numbers on the main diagonal is between 0 and 3).

$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix}$ doesnot not belong to F

$F' = \{(a, b, c, d) \mid 0 \le a+d \le 3\}$

$(1, 2, 0, 2) \in F'$ $\quad (1,2,0,2) + (0,2,3,3)$

$(0, 2, 3, 3) \in F'$ $\quad = (1,4,3,5)$ doesnot not belong in F' $\to$ violates axiom of subspace

$a+d > 6$

(ii) Convince me that $F = \{(a + 2b - c)x^3 + (2a + c)x^2 + (3a + 2b)x + (a + 2b - c)\}$ is a subspace of P_4 by writing F as span of independent polynomials.

$F' = [(a+2b-c, 2a+c, 3a+2b, a+2b-c)]$

$F' = \text{span}\{(1,2,3,1), (0,-1,-1,0)$

we go back to f

$F' = \text{span}\{(1,2,3,1), (2,0,2,2), (-1,1,0,-1)\}$

$F = \{x^3 + 2x^2 + 3x + 1, -x^2 - x\}$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 0 & 2 & 2 \\ -1 & 1 & 0 & -1 \end{bmatrix} \begin{matrix} -2R_1+R_2 \to R_2 \\ R_1+R_3 \to R_3 \end{matrix} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -4 & -4 & 0 \\ 0 & 3 & 3 & 0 \end{bmatrix} \tfrac{1}{4}R_2$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 3 & 3 & 0 \end{bmatrix} \xrightarrow{3R_2+R_3 \to R_3} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

QUESTION 4. (5 points) Let $L = \{a_3x^3 + a_2x^2 + a_1x + a_0 \in P_4 \mid \int_0^1 (a_3x^3 + a_2x^2 + a_1x + a_0)\, dx = 0\}$. Write L as span of independent polynomials (hence L is a subspace of P_4).

[Hint: Translate to R^4, i.e., What is the form of L in R^4?]

$\int_0^1 a_3x^3 = \frac{a_3x^4}{4}\Big|_0^1 = \frac{a_3}{4}$

$L' = \{(a_3, a_2, a_1, a_0) \in \mathbb{R}^4 \mid \frac{a_3}{4} + \frac{a_2}{3} + \frac{a_1}{2} + a_0 = 0\}$

$a_0 = -\frac{a_3}{4} - \frac{a_2}{3} - \frac{a_1}{2}$

$= \{(a_3, a_2, a_1, -\frac{a_3}{4} - \frac{a_2}{3} - \frac{a_1}{2}) \in \mathbb{R}^4 \quad a_1, a_2, a_3 \in \mathbb{R}\}$

$\text{span}\{(1,0,0,-\frac{1}{4}), (0,1,0,-\frac{1}{3}), (0,0,1,-\frac{1}{2})\}$

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{1}{4} \\ 0 & 1 & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & -1/2 \end{bmatrix}$$

$L = \text{span}\{(x^3 - \frac{1}{4}), (x^2 - 1/3), (x - 1/2)\}$

QUESTION 5. (10 points) Let $T : R^3 \to R^4$ be a linear transformation such that $T(1,1,1) = (0,1,-1,0)$, $T(-1,0,-1)$: $(0,2,-2,0)$, and $T(-2,-1,-1) = (0,0,-1,0)$.

(i) Find the standard matrix presentation of T.

$$\left[\begin{array}{ccc|cccc} 1 & 1 & 1 & 0 & 1 & -1 & 0 \\ -1 & 0 & -1 & 0 & 2 & -2 & 0 \\ -2 & -1 & -1 & 0 & 0 & -1 & 0 \end{array}\right] \begin{array}{l} R_1+R_2 \to R_2 \\ 2R_1+R_3 \to R_3 \end{array} \left[\begin{array}{ccc|cccc} 1 & 1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 3 & -3 & 0 \\ 0 & 1 & 1 & 0 & 2 & -3 & 0 \end{array}\right]$$

$$\begin{array}{l} -R_2+R_1 \to R_1 \\ -R_2+R_3 \to R_3 \end{array} \left[\begin{array}{ccc|cccc} 1 & 0 & 1 & 0 & -2 & 2 & 0 \\ 0 & 1 & 0 & 0 & 3 & -3 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 \end{array}\right] \quad -R_3+R_1 \to R_1$$

$$\left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 1 & 0 & 0 & 3 & -3 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 \end{array}\right] \qquad M = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 3 & -1 \\ 2 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(ii) Write Range(T) as span of independent points

$$\begin{bmatrix} 0 & 0 & 0 \\ -1 & 3 & -1 \\ 2 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \; 2R_2+R_3 \to R_3 \; \begin{bmatrix} 0 & 0 & 0 \\ -1 & 3 & -1 \\ 0 & 3 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\text{Rang}(T) = \text{span}\{(0,-1,2,0),(0,3,-3,0)\}$

(iii) Write Z(T) as span of independent points.

$$\begin{bmatrix} 0 & 0 & 0 \\ -1 & 3 & -1 \\ 2 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \; 2R_2+R_3 \to R_3 \; \begin{bmatrix} 0 & 0 & 0 \\ -1 & 3 & -1 \\ 0 & 3 & -2 \\ 0 & 0 & 0 \end{bmatrix} \; -R_3+R_2 \to R_2 \; \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 3 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$-a_1 + a_3 = 0$ $\quad$ a_3 free

$3a_2 - 2a_3 = 0$ $\quad$ a_1, a_2 leading

sol : $\{(a_3, \tfrac{2}{3}a_3, a_3) \mid a_3 \in \mathbb{R}\}$

span $\{(1, \tfrac{2}{3}, 1)\}$

dim(range)+dim(z)

$2 + 1$

$= 3$

= dim(domain)

to check

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QUESTION 6. (5 points) Find $|A|$ and $|2A|$, where A is 4×4 and $A \xrightarrow{-2R_3+R_1 \to R_1} B \xrightarrow{2R_4}$

$$C = \begin{bmatrix} 4 & 8 & 4 & -4 \\ -4 & -7 & -4 & 6 \\ -8 & -16 & -6 & 7 \\ -4 & -5 & -4 & 1 \end{bmatrix}.$$

$R_1+R_2 \to R_2$
$2R_1+R_3 \to R_3$
$R_1+R_4 \to R_4$

$$\begin{bmatrix} 4 & 8 & 4 & -4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 2 & -1 \\ 0 & 3 & 0 & -3 \end{bmatrix}$$

$-3R_2+R_4 \to R_4$

$$\begin{bmatrix} 4 & 8 & 4 & -4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & -9 \end{bmatrix}$$

$|C| = 4 \times 1 \times 2 \times -9$
$= -72$

$|C| = 2|B|$
$|B| = -36$

$|B| = |A| = -36$

$|2A| = 2^4 |A| = -576$

QUESTION 7. (4 points) Let $A = \begin{bmatrix} 2 & 4 & a \\ b & c & d \\ -2 & 3 & e \end{bmatrix}$. Given $|A| = 11$, Find $|B|$, where $B = \begin{bmatrix} 2 & 4 & a \\ b & c & d+3 \\ -2 & 3 & e \end{bmatrix}$. ~~Find $|B|$~~.

[Hint: Find $|A|$ and $|B|$ by definition. STARE well ...you might observe something]

$(-1)^{(1+1)}(2)\begin{vmatrix} c & d \\ 3 & e \end{vmatrix} + (-1)^{(1+2)} b \begin{vmatrix} 4 & a \\ 3 & e \end{vmatrix} + (-1)^{(1+3)} -2 \begin{vmatrix} 4 & a \\ c & d \end{vmatrix} = 11$

$2(ce-3d) - b(4e-3a) - 2(4d-ca) = 11$

$(-1)^{(1+1)} 2 \begin{vmatrix} c & d+3 \\ 3 & e \end{vmatrix} + (-1)^{(1+2)} b \begin{vmatrix} 4 & a \\ 3 & e \end{vmatrix} + (-1)^{(1+3)} -2 \begin{vmatrix} 4 & a \\ c & d+3 \end{vmatrix} = X$

$2(ce - 3(d+3)) - b(4e-3a) - 2(4(d+3) - ca) = X$

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$2ce - 6d - 4be + 3ab - 8d + ca = 11$
$2ce - 6d - 18 - 4be + 3ab - 8d - 24 + ca = X$

by elimenation

$18 + 24 = 11 - X$
$X = -31$

$|B| = -31$